

Renormalization In Coupled-Abelian Self-Dual Chern-Simons Models

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Abstract

An algebraic restriction of the nonabelian self-dual Chern-Simons-Higgs systems leads to coupled-abelian self-dual models with intricate mass spectra. The vacua are characterized by embeddings of $SU(2)$ into the gauge algebra; and in the broken phases, the gauge and real scalar masses are related to the exponents of the gauge algebra. In this paper we compute the gauge-gauge-Higgs couplings in the broken phases and use this to compute the finite renormalizations of the Chern-Simons coefficient in the various vacua.

Chern-Simons theories are well-known to exhibit interesting topological and renormalization properties. A Dirac-style quantum consistency condition leads to an integer quantization condition ($\kappa = \frac{1}{4\pi}(\text{integer})$) on the coefficient κ of the Chern-Simons term in the Lagrangian [1]. Such a consistency condition is remarkably robust under renormalization; for example, in an $SU(N)$ Chern-Simons-Yang-Mills theory, the bare Chern-Simons coefficient κ receives a (finite) additive renormalization shift which is an **integer** multiple of $\frac{1}{4\pi}$: $\kappa \rightarrow \kappa + \frac{N}{4\pi}$ [2]. When charged matter fields are coupled to the Chern-Simons gauge fields, this issue becomes more involved due to spontaneous symmetry breaking effects. Direct diagrammatic analyses

have led to the conclusion that when the symmetry is broken completely the diagrammatic shift in κ is some complicated noninteger (dimensionless) combination of the Chern-Simons mass scale and the Higgs mass scale [7]; but when the symmetry is only partially broken, leaving a residual nonabelian symmetry in the broken phase, this dimensionless shift for the broken generators is once again an integer multiple of $\frac{1}{4\pi}$ (with the integer being the Coxeter number of the residual nonabelian symmetry group) [9]. A deeper way to understand these results is through a generalization of the Coleman-Hill theorem [10] to incorporate symmetry breaking systems [8]. It is believed that the complicated shifts of κ derived in the diagrammatic approach actually reflect the appearance of gauge invariant terms in the effective action which mimic a Chern-Simons term at large distances as the Higgs field tends to its nonzero vacuum expectation value [7, 8]. This is an appealing and consistent picture, but it has only been explicitly demonstrated in abelian theories [8], not in nonabelian theories.

Another important feature of Chern-Simons theories is that they admit a self-dual form of matter-gauge theory. Such systems have a Bogomol'nyi lower bound for the energy, which is saturated by solutions to a set of first-order self-duality equations [3]. These systems have a special sixth-order Higgs potential (renormalizable in $2+1$ dimensions), and a consequence of the particular form of this potential is that the Chern-Simons mass scale is a multiple of the Higgs mass scale. In the abelian self-dual Chern-Simons system in the broken phase, the Higgs and gauge masses are in fact equal and a remarkable consequence of this is that the one-loop renormalization shift of the Chern-Simons parameter is just $\frac{1}{4\pi}$ [11]. In this case the source of the integer-quantized shift is the self-duality of the model, rather than the existence of a nonabelian symmetry in the broken phase. The purpose of this paper is to investigate whether or not such a quantized shift occurs in the more intricate coupled-

abelian models studied in [12], for which there are many inequivalent broken vacua, and for which the gauge and Higgs mass spectra reveal surprising degeneracy patterns. These coupled-abelian theories are based on a nonabelian gauge algebra \mathcal{G} , but are algebraically restricted (in a manner analogous to Toda theories) so that the coupling between the fields involves the Cartan matrix of \mathcal{G} . For simplicity we restrict our attention to simply-laced algebras, and we will often concentrate more specifically on $SU(N)$.

The nonabelian self-dual Chern-Simons-Higgs theory [4, 5, 6] is described by the following Lagrange density (in $2 + 1$ dimensional spacetime)

$$\mathcal{L} = -\text{tr} \left((D_\mu \phi)^\dagger D^\mu \phi \right) - \kappa \epsilon^{\mu\nu\rho} \text{tr} \left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) - V(\phi, \phi^\dagger) \quad (1)$$

where $V(\phi, \phi^\dagger)$ is the sixth-order self-dual potential

$$V(\phi, \phi^\dagger) = \frac{1}{4\kappa^2} \text{tr} \left\{ \left([[\phi, \phi^\dagger], \phi] - v^2 \phi \right)^\dagger \left([[\phi, \phi^\dagger], \phi] - v^2 \phi \right) \right\}. \quad (2)$$

The space-time metric is taken to be $g_{\mu\nu} = \text{diag}(-1, 1, 1)$, and we work with adjoint coupling so that the covariant derivative takes the form $D_\mu = \partial_\mu + [A_\mu, \]$. The trace runs over the

finite-dimensional representation of the compact simple Lie algebra \mathcal{G} to which the gauge field A_μ and the charged matter fields ϕ and ϕ^\dagger belong.

With the self-dual potential (2) there exists a Bogomol'nyi lower bound on the energy density which is saturated by solutions to the (relativistic) self-dual Chern-Simons equations

$$\begin{aligned} D_- \phi &= 0 \\ F_{+-} &= \frac{1}{\kappa^2} [(v^2 \phi - [[\phi, \phi^\dagger], \phi]), \phi^\dagger] \end{aligned} \quad (3)$$

where $D_- \equiv D_1 - iD_2$. Solutions to these self-duality equations fall into classes characterized by the asymptotic values of the scalar fields, corresponding to the gauge inequivalent vacua

of the self-dual potential (2). These (degenerate) vacua are determined by the algebraic condition

$$[[\phi, \phi^\dagger], \phi] = v^2 \phi \quad (4)$$

Identifying $\frac{1}{|v|}\phi$ with J_+ (and $\frac{1}{|v|}\phi^\dagger$ with J_-), this vacuum condition is equivalent to the $SU(2)$ commutation relations. Thus, the vacua are classified by the inequivalent embeddings of $SU(2)$ into the gauge algebra \mathcal{G} ; such embeddings are important in Lie algebra theory [13], and it is interesting to note that they have also featured prominently in other well-known self-dual gauge theories [14]. For $SU(N)$, the number of inequivalent vacua (including the trivial $\phi = 0$ one) is equal to the number, $p(N)$, of partitions of N .

The coupled-abelian models to be considered in this paper are obtained from (1,2,3) by restricting the fields according to the algebraic ansatz

$$\phi = \sum_{a=1}^r \phi^a E_a \quad A_\mu = i \sum_{a=1}^r A_\mu^a H_a \quad (5)$$

where r is the rank of \mathcal{G} . We work in a Chevalley basis for the Lie algebra, with H_a and E_a being the Cartan subalgebra and simple root step operators respectively. These generators satisfy the commutation and normalization conditions

$$\begin{aligned} [H_a, H_b] &= 0 & \text{tr}(H_a H_b) &= K_{ab} \\ [E_a, E_{-b}] &= \delta_{ab} H_b & \text{tr}(H_a E_b) &= 0 \\ [H_a, E_{\pm b}] &= \pm K_{ab} E_{\pm b} & \text{tr}(E_a E_{-b}) &= \delta_{ab} \end{aligned} \quad (6)$$

where $E_{-a} = E_a^\dagger$, and the Cartan matrix K expresses the inner products of the simple roots $\vec{\alpha}^{(a)}$ (normalized to have length $\sqrt{2}$)

$$K_{ab} = \vec{\alpha}^{(a)} \cdot \vec{\alpha}^{(b)} \quad (a, b = 1 \dots r) \quad (7)$$

As shown in [12], the algebraic restriction (5) is self-consistent. The self-dual potential (2) is now expressed in terms of r complex scalar fields ϕ^a

$$V = \frac{v^4}{4\kappa^2} \sum_{a=1}^r |\phi^a|^2 - \frac{v^2}{2\kappa^2} \sum_{a=1}^r \sum_{b=1}^r |\phi^a|^2 K_{ab} |\phi^b|^2 + \frac{1}{4\kappa^2} \sum_{a=1}^r \sum_{b=1}^r \sum_{c=1}^r |\phi^a|^2 K_{ab} |\phi^b|^2 K_{bc} |\phi^c|^2 \quad (8)$$

For $SU(2)$ this reduces to $V = \frac{1}{4\kappa^2} |\phi|^2 (2|\phi|^2 - v^2)^2$, which is just the self-dual abelian potential studied in [3], and which has two inequivalent vacua. For $SU(3)$, $K_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, and the self-dual potential is $V = \frac{1}{4\kappa^2} (4(|\phi^1|^6 + |\phi^2|^6) - 3(|\phi^1|^4 |\phi^2|^2 + |\phi^1|^2 |\phi^2|^4) - 4v^2(|\phi^1|^4 - |\phi^1|^2 |\phi^2|^2 + |\phi^2|^4) + v^4(|\phi^1|^2 + |\phi^2|^2))$. A contour plot of this function $V(\phi^1, \phi^2)$ is shown in Figure 1. Note the presence of the three inequivalent vacua: $(|\phi^1|, |\phi^2|) = (0, 0)$, $(|\phi^1|, |\phi^2|) = \frac{|v|}{\sqrt{2}}(1, 0)$, and $(|\phi^1|, |\phi^2|) = |v|(1, 1)$. [The minimum at $(|\phi^1|, |\phi^2|) = \frac{|v|}{\sqrt{2}}(0, 1)$ is equivalent to the one at $(|\phi^1|, |\phi^2|) = \frac{|v|}{\sqrt{2}}(1, 0)$.] We note that the algebraic restriction (5) is consistent with the vacuum condition (4), since a representative $\phi_{(0)}$ from each equivalence class of minima may be expanded as

$$\phi_{(0)} = |v| \sum_{a=1}^r \phi_{(0)}^a E_a \quad (9)$$

where the $\phi_{(0)}^a$ are some numerical coefficients. For example, the “maximal embedding vacuum” is defined by taking $(\phi_{(0)}^a)^2$ to be the coefficients of the decomposition of (half) the sum of positive roots, in terms of the simple roots: $\vec{\rho} \equiv \frac{1}{2} \sum_{\alpha > 0} \vec{\alpha} = \sum_{a=1}^r (\phi_{(0)}^a)^2 \vec{\alpha}^{(a)}$. For $SU(N)$ this means $\phi_{(0)}^a = \sqrt{a(N-a)/2}$, $a = 1 \dots N-1$. On the other hand, the simplest nontrivial vacuum solution to (4) is obtained by taking $\phi_{(0)}$ to involve just one step operator, which (up to equivalence) we can choose to be E_1 :

$$\phi_{(0)} = \frac{|v|}{\sqrt{2}} E_1 \quad (10)$$

All other vacua are also specified by a decomposition of the form in (9), and may be classified in terms of the Dynkin diagrams of the various subalgebras of \mathcal{G} [6]. Since these vacua are, in

turn, characterized by the maximal embedding vacua for the various subalgebras so obtained, we can limit our attention to the maximal embedding vacuum. It is a straightforward matter to pass to the other vacua.

Given $\phi_{(0)}$, the gauge and scalar mass spectra in that vacuum are determined by shifting $\phi \rightarrow \phi + \phi_{(0)}$, and keeping terms in the Lagrangian that are quadratic in A_μ and ϕ . The quadratic gauge part of the Lagrangian is

$$\mathcal{L}_{quad}^{gauge} = -\kappa \epsilon^{\mu\nu\rho} \text{tr}(\partial_\mu A_\nu A_\rho) - \text{tr}([\phi_{(0)}, A_\mu]^\dagger [\phi_{(0)}, A^\mu]) \quad (11)$$

Thus the gauge mass matrix is

$$M_{gauge} = \frac{1}{2\kappa} \left(ad(\phi_{(0)}^\dagger) ad(\phi_{(0)}) + ad(\phi_{(0)}) ad(\phi_{(0)}^\dagger) \right) = \frac{v^2}{2\kappa} (J_+ J_- + J_- J_+) = \frac{v^2}{2\kappa} \mathcal{C} \quad (12)$$

where \mathcal{C} is the $SU(2)$ quadratic Casimir for the particular embedding (we have also used the fact that $J_3 A \equiv 0$, which follows from the algebraic restriction of the gauge field to the Cartan subalgebra in (5)). For the maximal embedding case, the adjoint action of $SU(2)$ decomposes \mathcal{G} into r subblocks, of spin s_a , where the integers s_a are called the *exponents* of \mathcal{G} [13]. Thus, in the maximal embedding vacuum all r gauge fields acquire nonzero mass, with masses

$$m_a = m s_a (s_a + 1) \quad a = 1 \dots r, \quad (13)$$

where $m = \frac{v^2}{2\kappa}$ is a common mass scale, which is equal to the scalar mass in the unbroken phase. For $SU(N)$ the exponents are the integers: $1, 2, \dots, N-1$.

The scalar masses are obtained from the quadratic part of $V(\phi + \phi_{(0)})$. In the maximal embedding vacuum this leads to a mass (squared) matrix

$$M_{scalar}^2 = \frac{1}{\kappa^2} \left(ad(\phi_{(0)}^\dagger) ad(\phi_{(0)}) \right)^2 = m^2 \mathcal{C}^2 \quad (14)$$

where we have used the fact that $J_3\phi = \phi$, which follows from the algebraic decomposition of the ϕ field in (5). Hence, in the maximal embedding vacuum, there are r nonzero scalar masses, and these are *equal* to the gauge masses in (13).

These mass spectra refer to *classical* properties of the self-dual Chern-Simons system (1,5), and the degeneracy of the gauge and scalar masses is a reflection of the underlying $N = 2$ supersymmetry of these self-dual Chern-Simons models [15]. However, the masses are also important for quantum effects. In particular, consider the one-loop renormalization of the Chern-Simons coupling parameter κ . It is by now a standard diagrammatic computation to show that κ receives a finite additive shift (in Landau gauge) obtained from the zero (external) momentum limit of the parity-odd part of the gauge self-energy diagram shown in Figure 2.

The shift $\Delta\kappa$ is given by

$$\begin{aligned}
\Delta\kappa &= \frac{i}{2} \lim_{p^2 \rightarrow 0} \frac{p_\gamma}{p^2} \epsilon^{\gamma\mu\nu} \Pi_{\mu\nu}(p) \\
&= -\frac{i}{2\pi^3} \frac{\Gamma^2}{\kappa} \lim_{p^2 \rightarrow 0} \int d^3q \frac{p \cdot q}{p^2((p-q)^2 - m_s^2)(q^2 - m_g^2)} \\
&= \frac{1}{4\pi} \frac{4}{3} \frac{\Gamma^2}{\kappa} \frac{m_s + 2m_g}{(m_s + m_g)^2}
\end{aligned} \tag{15}$$

where Γ is the vertex factor giving the weight of the coupling between the gauge and scalar fields of mass m_g and m_s respectively. In the broken phase of the abelian system, the gauge and scalar masses are degenerate, and the gauge-gauge-scalar vertex factor is $\Gamma^2 = |\kappa|m_s = |\kappa|m_g$, so that $\Delta\kappa = \text{sign}(\kappa)\frac{1}{4\pi}$ [11]. To determine the analogous renormalization shifts for the coupled-abelian system with self-dual potential (8) one needs, in addition to the masses (13) of the gauge and scalar fields, the three-point gauge-gauge-scalar coupling factors, which

are determined by the cubic term

$$\text{tr} \left([\phi^\dagger, [A^\mu, \phi_{(0)}]] A_\mu^\dagger \right) + \text{tr} \left([\phi_{(0)}^\dagger, [A^\mu, \phi]] A_\mu^\dagger \right) \quad (16)$$

However, the fields A and ϕ must first be decomposed in terms of the basis of *physical* fields which diagonalize the respective mass matrices. This requires knowledge of not just the *eigenvalues* of the mass matrices, but also of the corresponding *eigenvectors*. This may be achieved by the following construction. Define an $r \times r$ diagonal matrix $D = \text{diag}(\phi_{(0)}^1, \phi_{(0)}^2, \dots, \phi_{(0)}^r)$, where the diagonal entries $\phi_{(0)}^a$ are the coefficients of the decomposition (9) of the vacuum solution $\phi_{(0)}$. (Note that we are considering the maximal embedding vacuum, so that all the $\phi_{(0)}^a$ are nonzero). It is then straightforward to check that the gauge and scalar mass matrices are given by

$$M_{gauge} = 2m D^2 K \quad M_{scalar}^2 = 4m^2 (D K D)^2 \quad (17)$$

where K is the Cartan matrix (7). Now define the eigenvectors $\vec{\lambda}^{(a)}$ and $\vec{\mu}^{(a)}$ of the matrices $D^2 K$ and $D K D$ respectively:

$$(D^2 K) \vec{\lambda}^{(a)} = \frac{1}{2} s_a (s_a + 1) \vec{\lambda}^{(a)} \quad (D K D) \vec{\mu}^{(a)} = \frac{1}{2} s_a (s_a + 1) \vec{\mu}^{(a)} \quad (18)$$

These eigenvectors can be normalized as

$$\vec{\lambda}^{(a)T} K \vec{\lambda}^{(b)} = \delta^{ab} \quad \vec{\mu}^{(a)T} \vec{\mu}^{(b)} = \delta^{ab} \quad (19)$$

Given these eigenvectors, we now decompose the gauge and scalar fields A and ϕ as (we suppress the spacetime index on the gauge field as it is not important here)

$$A = \frac{i}{\sqrt{2}} \sum_{a=1}^r A_{phys}^a (\vec{\lambda}^{(a)} \cdot \vec{H})$$

$$\phi = \frac{1}{\sqrt{2}} \sum_{a=1}^r \phi_{phys}^a (\vec{\mu}^{(a)} \cdot \vec{E}) \quad (20)$$

Then the fields A_{phys}^a and ϕ_{phys}^a are *real, physical* fields, with masses $m_a = m s_a (s_a + 1)$. In terms of this physical field decomposition, the cubic term (16) becomes

$$\frac{|v|}{2\sqrt{2}} \phi_{phys}^a A_{phys}^b A_{phys}^c \left(\sqrt{s_b(s_b + 1)s_c(s_c + 1)} \sum_{d=1}^r \frac{\mu_d^{(a)} \mu_d^{(b)} \mu_d^{(c)}}{\phi_{(0)}^d} \right) \quad (21)$$

In general, these three-point coupling terms are nonzero. Therefore, in the maximally broken phase the coupled-abelian systems involve interactions between gauge and scalar fields of different masses. All possible couplings must be taken into account and the renormalization shifts for given external A_{phys}^a fields may be computed from (15) using (13) for the masses and (21) for the vertex factors. (Note that the normalizations are such that the restricted $SU(2)$ case is a single abelian system.) The shifts $\Delta\kappa$ for $SU(2)$, $SU(3)$, $SU(4)$, and $SU(5)$ are presented in Table 1 for all the various inequivalent vacua, and for the various physical gauge field components A_{phys}^a . Certain patterns are clear from these results. First, in general the renormalization shift is *not* an integer multiple of $\frac{1}{4\pi}$. Second, one can recognize from the various vacua the contributions corresponding to the maximal embedding vacuum of the lower subalgebras. The only vacua for which the renormalization shift *is* a multiple of $\frac{1}{4\pi}$ are those which correspond to disconnected and isolated $SU(2)$ components. In these vacua the model reduces (at cubic level) to a set of independent abelian models, for each of which the shift is known to be $\frac{1}{4\pi}$ owing to the degeneracy of the gauge and Higgs masses, which is in turn due to the self-dual nature of the system.

These results indicate that (unlike in the abelian self-dual theory [11]) the self-duality of the coupled-abelian model is not, in itself, sufficient to produce integer multiples of $\frac{1}{4\pi}$ for the one-loop renormalization shift of the Chern-Simons coefficient κ . Several aspects deserve

to be investigated further. First, it would be interesting to analyse these models, which in a sense lie between the abelian and nonabelian cases, using the effective action techniques which have been applied to abelian Chern-Simons-Higgs theories [8]. Presumably the various zero-momentum parity-odd contributions to the gauge self-energy computed using (15) can be interpreted as coming from a term in the effective action that mimics a Chern-Simons term at large distance. Second, the role of fermionic fields in a supersymmetric realization would also be of interest. Finally, these results should be extended to the fully nonabelian self-dual system.

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Algebra	Vacuum $\phi_{(0)}$	A_{phys}^1	A_{phys}^2	A_{phys}^3	A_{phys}^4
$SU(2)$	$\frac{1}{\sqrt{2}}E_1$	1			
$SU(3)$	$\frac{1}{\sqrt{2}}E_1$	0	1		
	$E_1 + E_2$	$\frac{1}{2}$	$\frac{13}{8}$		
$SU(4)$	$\frac{1}{\sqrt{2}}E_1$	0	0	1	
	$\frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3$	0	1	1	
	$E_1 + E_2$	0	$\frac{1}{2}$	$\frac{13}{8}$	
	$\sqrt{\frac{3}{2}}E_1 + \sqrt{2}E_2 + \sqrt{\frac{3}{2}}E_3$	$\frac{3}{10}$	$\frac{115}{108}$	$\frac{933}{490}$	
$SU(5)$	$\frac{1}{\sqrt{2}}E_1$	0	0	0	1
	$\frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3$	0	0	1	1
	$E_1 + E_2$	0	0	$\frac{1}{2}$	$\frac{13}{8}$
	$\frac{1}{\sqrt{2}}E_1 + E_3 + E_4$	0	$\frac{1}{2}$	1	$\frac{13}{8}$
	$\sqrt{\frac{3}{2}}E_1 + \sqrt{2}E_2 + \sqrt{\frac{3}{2}}E_3$	0	$\frac{3}{10}$	$\frac{115}{108}$	$\frac{933}{490}$
	$\sqrt{2}E_1 + \sqrt{3}(E_2 + E_3) + \sqrt{2}E_4$	$\frac{1}{5}$	$\frac{4549}{6048}$	$\frac{118553}{82810}$	$\frac{1488125}{731808}$

Table 1: The additive renormalization shifts $\Delta\kappa$, in units of $\frac{1}{4\pi}sign(\kappa)$, for the various inequivalent nontrivial vacua $\phi_{(0)}$ of $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$, for the various physical components A_{phys}^a of the gauge field.

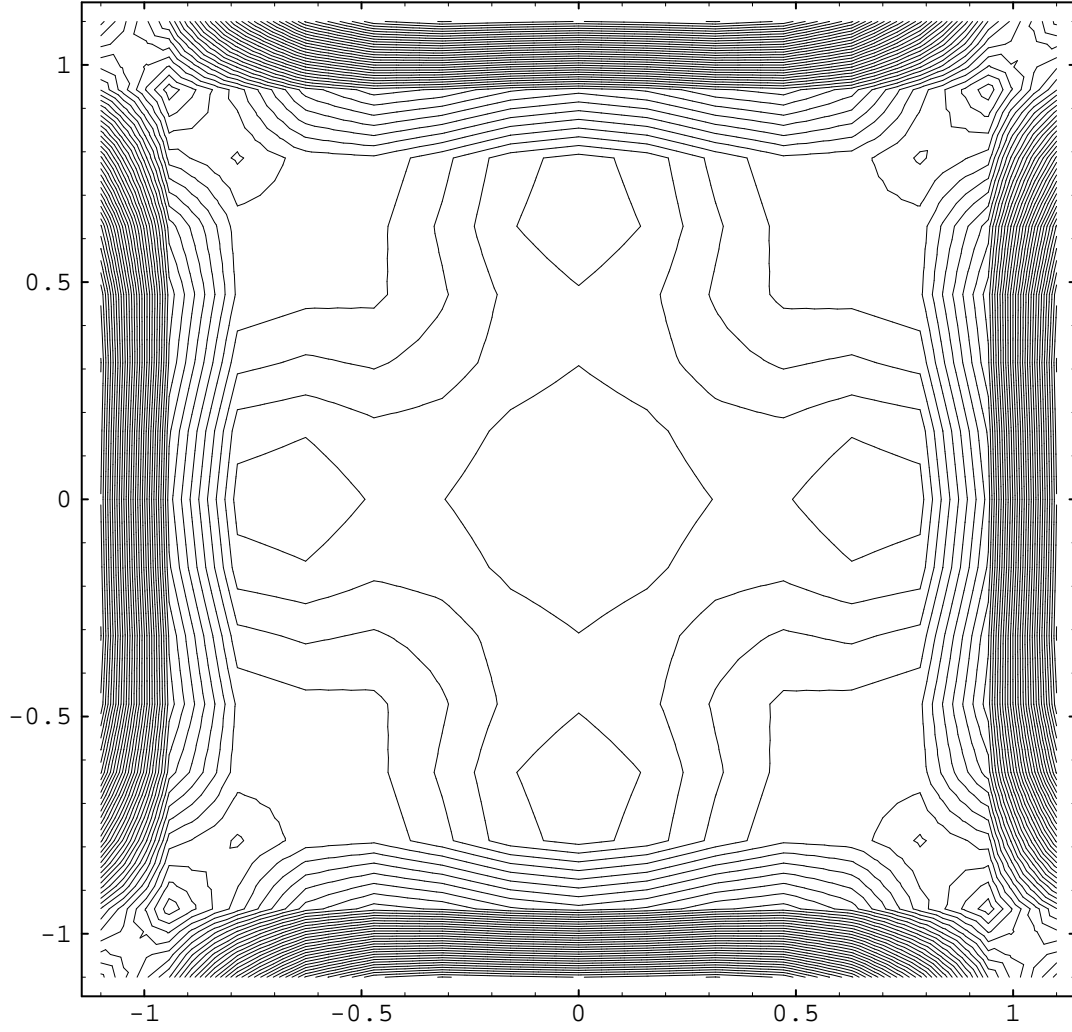


Figure 1: Contour plot of the self-dual potential $V(\phi^1, \phi^2)$ for the restricted $SU(3)$ model. Note the three classes of inequivalent vacua: (i) at $(\phi^1, \phi^2) = (0, 0)$; (ii) at $(\phi^1, \phi^2) = (\pm 1/\sqrt{2}, 0)$ and $(0, \pm 1/\sqrt{2})$; and (iii) at $(\phi^1, \phi^2) = (\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$.

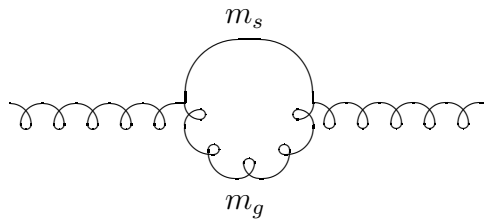


Figure 2: Gauge self-energy arising from gauge-scalar interaction.

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